Lecture 14:

Exit Times

Ex1. (Two-year college) Last time, we introduced a Markov chain of a two-year college model, with transition matrix

$$P = \begin{array}{c|cccc} & 1 & 2 & G & D \\ \hline 1 & 0.25 & 0.6 & 0 & 0.15 \\ 2 & 0 & 0.2 & 0.7 & 0.1 \\ G & 0 & 0 & 1 & 0 \\ D & 0 & 0 & 0 & 1 \end{array}$$

Q: On the average, how many years does a student take to graduate or dropout?

A: Let $g(x)$ be the expected time for a student to stay in the college, starting from the state x.

Then $g(G) = g(D) = 0$. By considering what happens in one step:

$$\begin{cases} g(1) = 1 + 0.25\, g(1) + 0.6\, g(2) + 0.15\, g(D) \\ g(2) = 1 + 0.2\, g(2) + 0.7\, g(G) + 0.1\, g(D) \end{cases}$$

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Solving the system of equations gives $g(1) = 2.3333$ and $g(2) = 1.25$.

Ex2 (Tennis) Last time, we introduced the tennis game that if a player wins each point with probability 0.6, then the Markov chain, in which the state is the difference of the scores, has the following transition matrix

$$P = \begin{array}{c} \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccccc} -2 & -1 & 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.4 & 0 & 0.6 & 0 & 0 \\ 0 & 0.4 & 0 & 0.6 & 0 \\ 0 & 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

Let $g(x)$ be the expected time to complete the game when the current state is x. Let $T = \min\{\tau_2, \tau_{-2}\}$, then

$$
\begin{aligned}
g(x) &= \mathbb{E}_x T \\
&= \sum_{k=1}^{\infty} k \cdot \mathbb{P}_x(T=k) . \\
&= \sum_{k=1}^{\infty} k \cdot \sum_{y \in \chi} \mathbb{P}(T=k, X_1=y \mid X_0=x) \\
&= \sum_{k=1}^{\infty} \sum_{y \in \chi} k \cdot \mathbb{P}(T=k \mid X_1=y, X_0=x) P_{xy} \\
&= \sum_{k=1}^{\infty} \sum_{y \in \chi} k \cdot \mathbb{P}(T=k \mid X_1=y) P_{xy} \\
&= \sum_{y \in \chi} P_{xy} \sum_{k=1}^{\infty} k \cdot \mathbb{P}(T=k \mid X_1=y) \\
&= \sum_{\substack{y \in \chi \\ y \notin \{-2,2\}}} P_{xy} \sum_{k=1}^{\infty} k \cdot \mathbb{P}(T=k \mid X_1=y) \\
&\quad + \sum_{y \in \{2,-2\}} P_{xy} \cdot (1 + 0 + 0 + \cdots) \\
&= \sum_{\substack{y \in \chi \\ y \notin \{-2,2\}}} P_{xy} \sum_{k=1}^{\infty} k \cdot \mathbb{P}(T=k-1 \mid X_0=y) + \sum_{y \in \{2,-2\}} P_{xy} \\
&= \sum_{\substack{y \in \chi \\ y \notin \{-2,2\}}} P_{xy} \sum_{l=0}^{\infty} (l+1) \mathbb{P}_y(T=l) + \sum_{y \in \{2,-2\}} P_{xy} \\
&= \sum_{\substack{y \in \chi \\ y \notin \{-2,2\}}} P_{xy} \left(\sum_{l=0}^{\infty} l \, \mathbb{P}_y(T=l) + \sum_{l=0}^{\infty} \mathbb{P}_y(T=l) \right) + \sum_{y \in \{2,-2\}} P_{xy} \\
&= \sum_{\substack{y \in \chi \\ y \notin \{-2,2\}}} P_{xy} \left(g(y) + 1 \right) + \sum_{y \in \{2,-2\}} P_{xy} \cdot (\underbrace{g(y)}_{\text{“}=0\text{”}} + 1)
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{l=0}^{\infty} \mathbb{P}_y(T=l) \\
&= \mathbb{P}_y(T<\infty) + \mathbb{P}_y(T=\infty) \\
&= 1 .
\end{aligned}
$$

$$= \sum_{y \in \mathcal{X}} P_{xy} \left(g(y) + 1 \right)$$

$$= \sum_{y \in \mathcal{X}} P_{xy} \cdot g(y) + \sum_{y \in \mathcal{X}} P_{xy}$$

$$= \sum_{y \in \mathcal{X}} P_{xy} \cdot g(y) + 1 .$$

Notice that $g(2) = g(-2) = 0$. Plugging the values of P_{xy}, $g(2)$, and $g(-2)$ yields

$$\begin{cases} g(-1) = 1 + 0.6\, g(0) ; \\ g(0) = 1 + 0.4\, g(-1) + 0.6\, g(1) ; \\ g(1) = 1 + 0.4\, g(0) . \end{cases}$$

Thus, $g(-1) = \frac{43}{13}$, $g(0) = \frac{50}{13}$, $g(1) = \frac{33}{13}$.

Remark 14.1. If we let R be the restriction of P on $C = \{-1, 0, 1\}$, then $g(x) = 1 + \sum_{y \in C} R_{xy} \cdot g(y)$ $\forall y \in C$.

Let $\vec{g} = (g(-1), g(0), g(1))^T$ and $\vec{1} = (1, 1, 1)^T$, then $(I - R)\vec{g} = \vec{1}$. Notice that the matrix $I - R$ is invertible. Thus, $\vec{g} = (I - R)^{-1} \cdot \vec{1}$.

Theorem 14.1. (Exit Time) Let $T = \min\{n \geq 0 : X_n \in A\}$ be the time to exit. Suppose $C = \mathcal{X} \setminus A$ is finite and $\mathbb{P}_x(T < \infty) > 0$ $\forall x \in C$. If $g(a) = 0$ $\forall a \in A$ and

$$g(x) = 1 + \sum_{y \in \mathcal{X}} P_{xy} \cdot g(y) \quad \forall x \in C, \text{ then } g(x) = \mathbb{E}_x T.$$

Proof.

$$\mathbb{E}_x\, g(X_T)$$
$$= \mathbb{E}_x\Big[g(X_0) + \sum_{k=0}^{\infty}\big(g(X_{k+1}) - g(X_k)\big) \cdot \mathbb{1}_{\{T > k\}}\Big]$$
$$= \underbrace{\mathbb{E}_x\, g(X_0)}_{g(x)} + \sum_{k=0}^{\infty} \mathbb{E}_x\Big[\big(g(X_{k+1}) - g(X_k)\big) \cdot \mathbb{1}_{\{T > k\}}\Big]$$

For each fixed $k \in \mathbb{N}$, let $G_k = (X_0 = x, X_1, \cdots, X_k)$.

Then, by the tower rule,

$$\mathbb{E}_x\Big[\big(g(X_{k+1}) - g(X_k)\big) \cdot \mathbb{1}_{\{T > k\}}\Big]$$
$$= \mathbb{E}_x\Big[\mathbb{E}\Big[\big(g(X_{k+1}) - g(X_k)\big) \cdot \mathbb{1}_{\{T > k\}} \,\Big|\, G_k\Big]\Big]$$
$$= \mathbb{E}_x\Big[\mathbb{E}[g(X_{k+1}) - g(X_k) \mid G_k] \cdot \mathbb{E}[\mathbb{1}_{\{T > k\}} \mid G_k]\Big]$$
$$= \mathbb{E}_x\Big[\big(\mathbb{E}[g(X_{k+1}) \mid G_k] - g(X_k)\big) \cdot \mathbb{E}[\mathbb{1}_{\{T > k\}} \mid G_k]\Big]$$

$= \mathbb{E}_x \left[\left(\mathbb{E}[g(X_{k+1}) \mid X_k] - g(X_k) \right) \cdot \mathbb{E}[\mathbb{1}_{\{T>k\}} \mid \mathcal{G}_k] \right]$

$= \mathbb{E}_x \left[\left(\sum_{y \in \mathcal{X}} g(y) \cdot \mathbb{P}_{X_k, y} - g(X_k) \right) \cdot \mathbb{E}[\mathbb{1}_{\{T>k\}} \mid \mathcal{G}_k] \right]$

$= \mathbb{E}_x \left[\left(\{ g(X_k) - 1 \} - g(X_k) \right) \cdot \mathbb{E}[\mathbb{1}_{\{T>k\}} \mid \mathcal{G}_k] \right]$

$= - \mathbb{E}_x [\mathbb{E}[\mathbb{1}_{\{T>k\}} \mid \mathcal{G}_k]]$

$= - \mathbb{E}_x [\mathbb{1}_{\{T>k\}}]$

$= - [1 \cdot \mathbb{P}_x(T>k) + 0 \cdot \mathbb{P}_x(T \le k)]$

$= - \mathbb{P}_x(T>k)$.

Therefore, $\mathbb{E}_x g(X_T) = g(x) - \sum_{k=0}^{\infty} \mathbb{P}_x(T>k) = g(x) - \mathbb{E}_x T$.

On the other hand, $\mathbb{E}_x g(X_T) = \mathbb{E}_x 0 = 0$, since $X_T \in A$.

This implies $g(x) = \mathbb{E}_x T$. $\square$

Ex 3. (Duration of a fair game). Consider the gambler's ruin with probability $p = \frac{1}{2}$. Let $T = \min\{n \ge 0 : X_n \in \{0, N\}\}$.

We claim that $\mathbb{E}_x T = x(N-x)$.

①. Verify the guess. Notice that $C = \{1, 2, \cdots, N-1\}$ is finite, $\mathbb{P}_x(T < \infty) > (\frac{1}{2})^x > 0 \ \forall x \in C$, and $g(0) = g(N) = 0$. Thus, by Theorem 14.1, the linear system of equations $g(x) = 1 + \sum_{y \in \mathcal{X}} P_{xy}\, g(y)$ has a unique solution $g(x) = \mathbb{E}_x T$.

To verify the guess, plug in $g(x) = x(N-x)$ into the formula and the right hand side gives

$$1 + \tfrac{1}{2} g(x-1) + \tfrac{1}{2} g(x+1)$$

$$= 1 + \tfrac{1}{2}(x-1)(N-x+1) + \tfrac{1}{2}(x+1)(N-x-1)$$

$$= x(N-x) = LHS.$$

②. How to derive the answer?

From $g(x) = 1 + \frac{1}{2} g(x+1) + \frac{1}{2} g(x-1) \ \forall \ 0 < x < N$.

$$\underbrace{g(x+1) - g(x)}_{b_{x+1}} = -2 + \underbrace{g(x) - g(x-1)}_{b_x}.$$

$$\Rightarrow b_{x+1} = -2 + b_x = -4 + b_{x-1} = -6 + b_{x-2} = \cdots$$

$\Rightarrow b_{x+1} = b_1 - 2x$, where $b_1 = g(1) - g(0)$

$\Rightarrow g(x+1) - g(x) = b_1 - 2x, \quad \forall\ 0 \le x < N.$

Since $0 = g(N) - g(0) = \sum_{x=0}^{N-1} g(x+1) - g(x)$

$$= \sum_{x=0}^{N-1} (b_1 - 2x)$$

$$= b_1 \cdot N - 2 \cdot \frac{N(N-1)}{2}$$

$$= b_1 N - N(N-1).$$

$\Rightarrow b_1 = N-1.$

$\Rightarrow g(x+1) - g(x) = N-1-2x, \quad \forall\ 0 \le x \le N-1.$

$\Rightarrow g(x) = g(x) - g(0)$

$$= \sum_{k=0}^{x-1} g(k+1) - g(k).$$

$$= \sum_{k=0}^{x-1} (N-1) - 2k$$

$$= (N-1)x - 2 \cdot \frac{x(x-1)}{2}$$

$$= (N-1)x - x(x-1).$$

$$= x(N-x).$$

Ex 4. (Duration of Nonfair Game) Consider a gambler's ruin chain where $P_{i,i+1} = p$ and $P_{i,i-1} = q = 1-p$, $\forall i = 1, 2, \cdots, N-1$. Let $T = \min\{n \geq 0 : X_n \in \{0, N\}\}$.

We claim that

$$\mathbb{E}_x T = \frac{x}{q-p} - \frac{N}{q-p} \cdot \frac{1-\left(\frac{q}{p}\right)^x}{1-\left(\frac{q}{p}\right)^N}.$$

① Verify the guess

$$\begin{aligned}
RHS &= 1 + p\,g(x+1) + q\,g(x-1) \\
&= 1 + p \cdot \frac{x+1}{q-p} - \frac{N}{q-p} \cdot \frac{p\left(1-\left(\frac{q}{p}\right)^{x+1}\right)}{1-\left(\frac{q}{p}\right)^N} \\
&\quad + q \cdot \frac{x-1}{q-p} - \frac{N}{q-p} \cdot \frac{q\left(1-\left(\frac{q}{p}\right)^{x-1}\right)}{1-\left(\frac{q}{p}\right)^N} \\
&= 1 + \frac{(p+q)x + p - q}{q-p} - \frac{N}{q-p} \cdot \frac{\left(1-\left(\frac{q}{p}\right)^x\right) \cdot (q+p)}{1-\left(\frac{q}{p}\right)^N} \\
&= \frac{x}{q-p} - \frac{N}{q-p} \cdot \frac{1-\left(\frac{q}{p}\right)^x}{1-\left(\frac{q}{p}\right)^N} \\
&= LHS.
\end{aligned}$$

② To derive the formula:

$$g(x) = 1 + p\,g(x+1) + (1-p)\,g(x-1), \quad \forall\ 0<x<N.$$

$$p(\underbrace{g(x+1)-g(x)}_{b_{x+1}}) = -1 + (1-p)(\underbrace{g(x)-g(x-1)}_{b_x})$$

Let $\theta = \frac{1-p}{p}$, then

$$b_{x+1} = -\frac{1}{p} + \frac{1-p}{p}\cdot b_x$$

$$= -\frac{1}{p} + \theta\cdot b_x$$

$$= -\frac{1}{p}(1+\theta) + \theta^2 b_{x-1}$$

$$= \cdots$$

$$= -\frac{1}{p}(1+\theta+\theta^2+\cdots+\theta^{x-1}) + \theta^x\cdot b_1$$

$$= -\frac{1-\theta^x}{p(1-\theta)} + \theta^x b_1.$$

$$\Rightarrow g(x+1) - g(x) = \theta^x\cdot\left(b_1 + \frac{1}{p-q}\right) - \frac{1}{p-q}, \quad 0\le x<N.$$

$$\Rightarrow 0 = g(N) - g(0)$$

$$= \sum_{k=0}^{N-1} g(k+1) - g(k)$$

$$= \sum_{k=0}^{N-1} \theta^k\cdot\left(b_1 + \frac{1}{p-q}\right) - \frac{1}{p-q}$$

$$= \frac{-N}{p-q} + \left(b_1 + \frac{1}{p-q}\right) \cdot \frac{1-\theta^N}{1-\theta}.$$

$$\Rightarrow b_1 = -\frac{1}{p-q} \cdot \frac{N}{p-q} \cdot \frac{1-\theta}{1-\theta^N}.$$

Thus, $g(x) = g(x) - g(0)$

$$= \sum_{k=0}^{x-1} g(k+1) - g(k)$$

$$= \frac{-x}{p-q} + \left(b_1 + \frac{1}{p-q}\right) \cdot \frac{1-\theta^x}{1-\theta}$$

$$= \frac{-x}{p-q} + \frac{N}{p-q} \cdot \frac{1-\theta^x}{1-\theta^N}.$$

This is the end of this lecture!